Computing Bounds on the Geometrical Quality of Curvilinear Finite Elements

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Gmsh Workshop 2013

- Introduction
- Bounds for Geometrical Validity
- Bounds on Geometrical Quality

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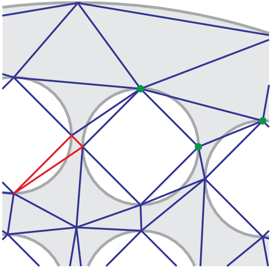
Way of Building a Curvilinear Mesh



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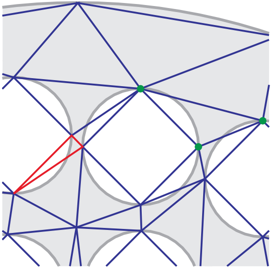
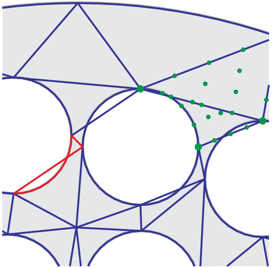


→poor quality elements

→invalid elements

Mesh entities classified on the boundaries are curved



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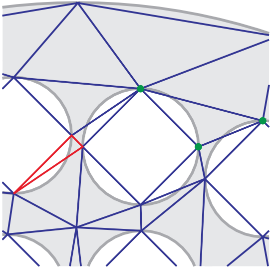

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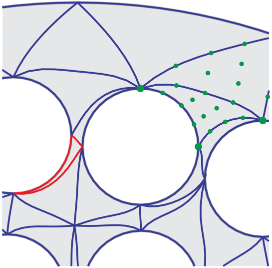

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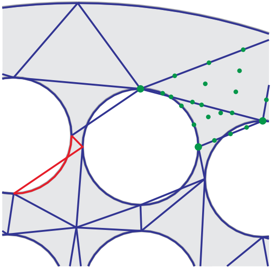
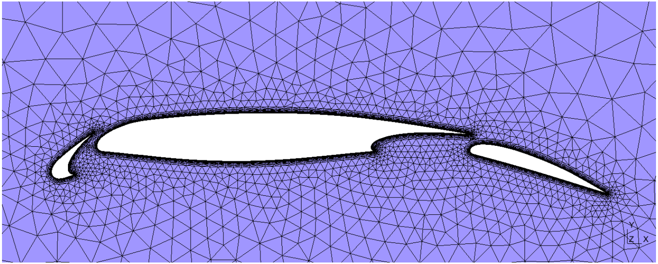
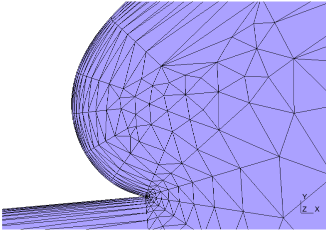


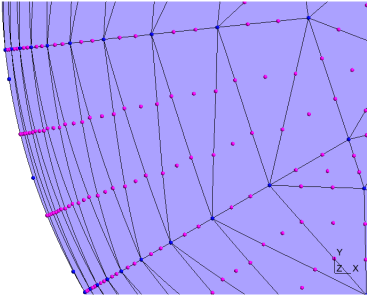
Optimization

Computationally expensive !

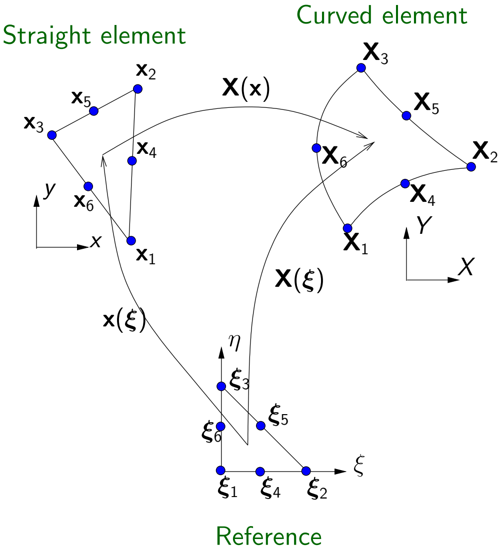
Example of Optimization: Wing (3-order Triangles)







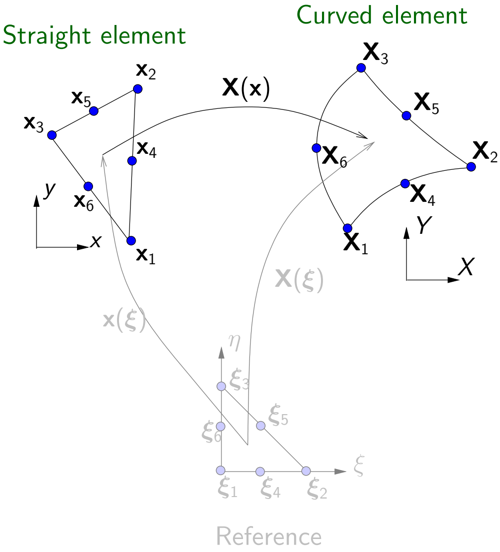
Quality Deterioration Control



Quality Deterioration Control

Straight element
→ good quality

Mapping $\mathbf{X}(\mathbf{x})$
⇒ distortion that should be controlled



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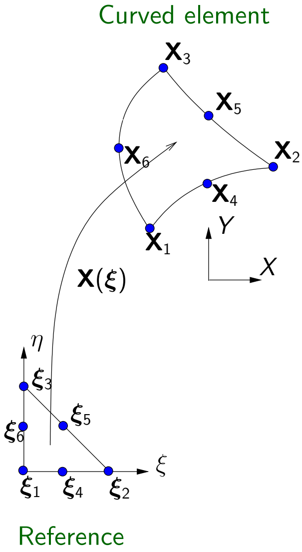
Element Validity

An element is valid...

if its mapping $\mathbf{X}(\boldsymbol{\xi})$ is bijective



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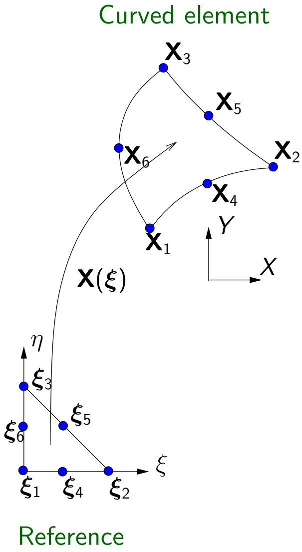
i.e. if $\boxed{J(\boldsymbol{\xi}) := \det \mathbf{X}_{,\boldsymbol{\xi}} > 0}$

Jacobian determinant $J(\boldsymbol{\xi})$



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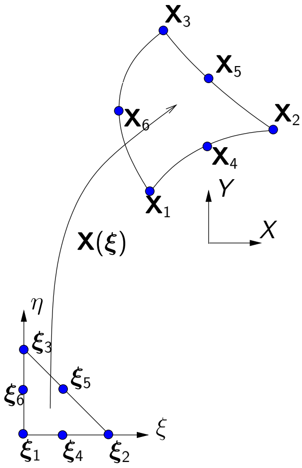
Necessary and sufficient condition :

$$J_{\min} = \min_{\xi} J(\boldsymbol{\xi}) > 0$$

Curved element



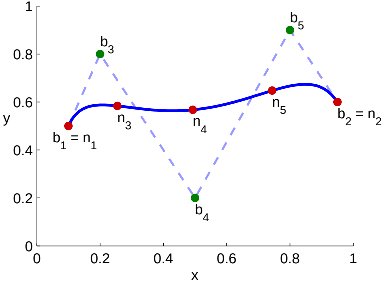
Reference

Use of Bézier for Computing Bounds on $J_{\min}$ - Arbitrary Element Case

$$J(\boldsymbol{\xi}) = \sum_{i=1}^{N} b_i \, B_i(\boldsymbol{\xi})$$

Bézier functions $B_i(\boldsymbol{\xi})$

Bézier coefficients b_i



Property: $J(\boldsymbol{\xi}) \in \text{conv}(b_1, b_2, \ldots, b_N)$

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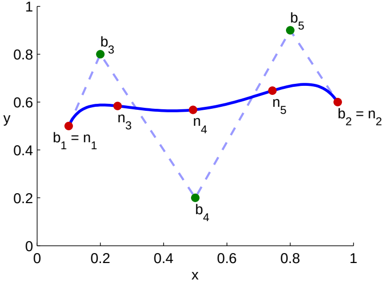

Property: $J(\boldsymbol{\xi}) \in \mathrm{conv}(b_1, b_2, \ldots, b_N)$

Lower bound :

$$\mathbf{b_{min}} = \min(\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_N}) \leq \mathbf{J_{min}}$$

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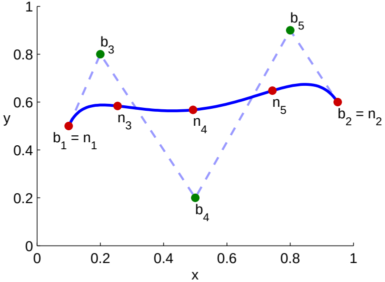


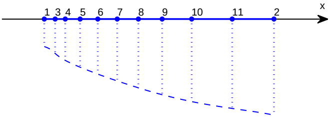
Property: $J(\boldsymbol{\xi}) \in \text{conv}(b_1, b_2, \ldots, b_N)$

Also: at the corners, $b_k = J(\boldsymbol{\xi}_k) \quad (k = 1, \ldots, N_{\text{corner}})$

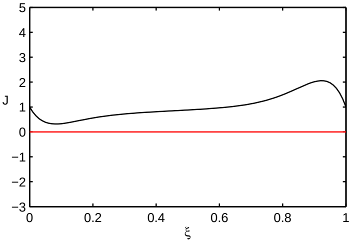
Lower and upper bound :

$$\mathbf{b_{min}} = \min(\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_N}) \leq \mathbf{J_{min}} \leq \min(\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_{N_{corner}}}) = \mathbf{c_{min}}$$

Accurate Bounds

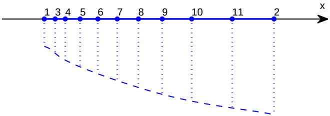


Element Geometry



Jacobian $J(\xi)$

Accurate Bounds



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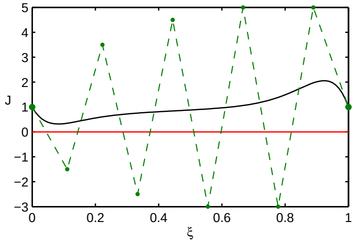
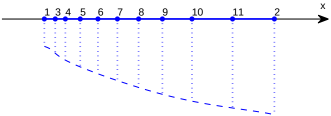


Jacobian $J(\xi)$

$J_{\min} \in$ 1. [-3,1]

Accurate Bounds



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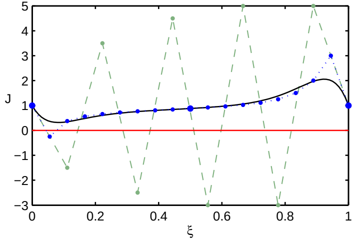


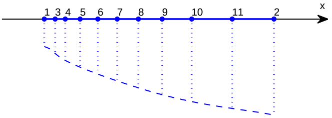
Jacobian $J(\xi)$

Subdivision = Restriction to Q subdomains

1. [-3,1]

$J_{\min} \in$ 2. [-0.25, 0.88] [0.88, 0.88]

Accurate Bounds



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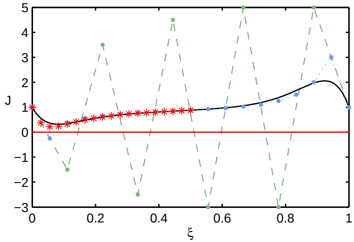
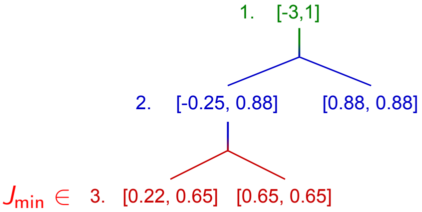


Jacobian $J(\xi)$

Subdivision = Restriction to Q subdomains

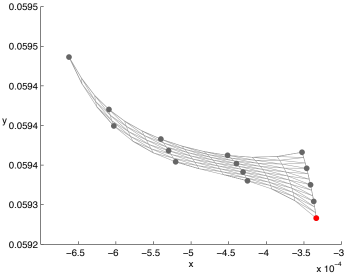
Each subdivision provides better bounds :

$$\begin{cases} b_{\min}^{\text{sub}} \geq b_{\min} \\ c_{\min}^{\text{sub}} \leq c_{\min} \end{cases}$$

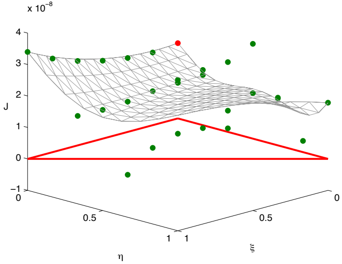


Application : 4-order Triangle

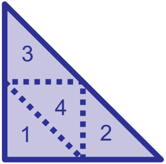
Element Geometry



Jacobian



Subdomains :



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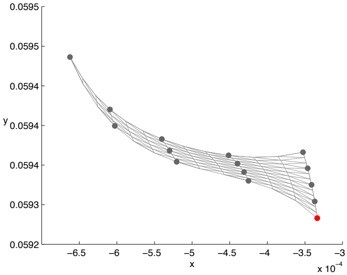
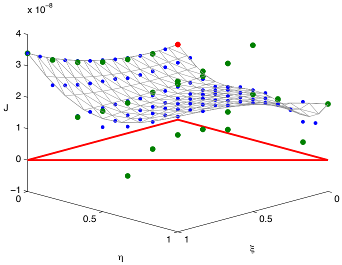


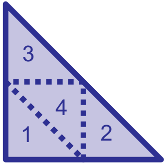
Jacobian



Subdivision

- adaptative scheme
- quadratic convergence

Subdomains :



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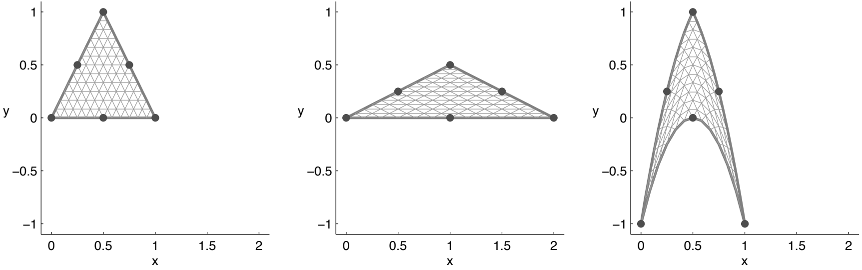
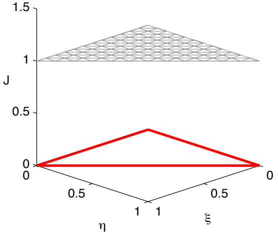
Jacobian Determinant is Not Enough for Optimization

Jacobian Determinant captures only **area changes !**



The three triangles have

- same total area
- same constant Jacobian

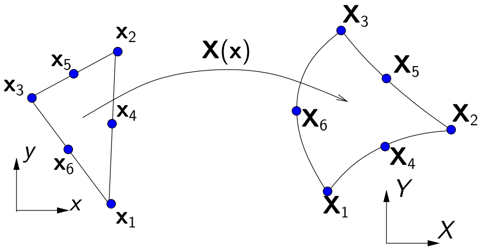


Measure of Variations of Lengths

Metric tensor of the mapping:

$$\boxed{\mathbf{M} := \mathbf{J}^T\mathbf{J}}$$

Jacobian matrix $\mathbf{J}$



Ratio between deformed and undeformed lengths:

$$r = \frac{||\mathbf{J}\mathbf{v}||}{||\mathbf{v}||} = \sqrt{\frac{\mathbf{v}^T\mathbf{M}\mathbf{v}}{\mathbf{v}^T\mathbf{v}}}$$

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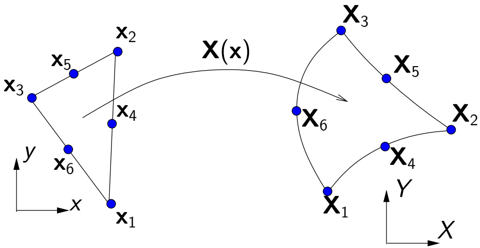

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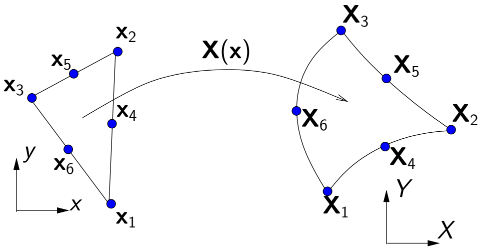
$$r = \frac{||\mathbf{J}\mathbf{v}||}{||\mathbf{v}||} = \sqrt{\frac{\mathbf{v}^T\mathbf{M}\mathbf{v}}{\mathbf{v}^T\mathbf{v}}} \quad \Rightarrow \quad r_{\min} = \sqrt{\lambda_{\min}(\mathbf{M})}$$

eigenvalue λ

and resp. for the max

Eigenvalues of Metric Tensor - in 2D

$$\mathbf{M} = \begin{pmatrix} ||\mathbf{X}_{,x}||^2 & \mathbf{X}_{,x} \cdot \mathbf{X}_{,y} \\ \mathbf{X}_{,x} \cdot \mathbf{X}_{,y} & ||\mathbf{X}_{,y}||^2 \end{pmatrix}$$



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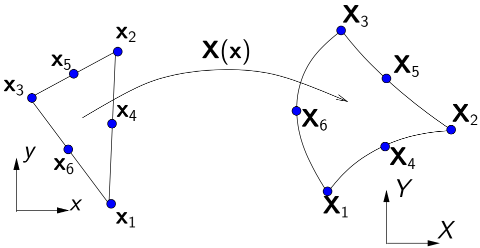

$$\lambda(\mathbf{M}) = \frac{1}{2}\left[||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,y}||^2 \pm \sqrt{(||\mathbf{X}_{,x}||^2 - ||\mathbf{X}_{,y}||^2)^2 + (2\ \mathbf{X}_{,x} \cdot \mathbf{X}_{,y})^2}\right]$$

Eigenvalues of Metric Tensor - in 2D

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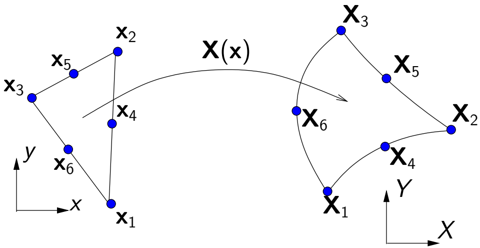

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$$\Rightarrow \boxed{\lambda(\mathbf{M}) = \frac{1}{2}\left[q \pm \sqrt{s^2 + t^2} \right]}$$

$q(\xi)$, $s(\xi)$, $t(\xi)$:

- polynomial functions
- same order as $J(\xi)$

Operator definition

For
- polynomial function f,
- exact Bézier expansion after k subdivisions,
- subdomain $d \in D(k)$:

$\underset{[d]}{\mathrm{minb}}\, f(\cdot)$ & $\underset{[d]}{\mathrm{maxb}}\, f(\cdot)$:= min/max of Bézier coefficients

$\underset{[d]}{\mathrm{minl}}\, f(\cdot)$ & $\underset{[d]}{\mathrm{maxl}}\, f(\cdot)$:= min/max of Bézier coefficients at corners

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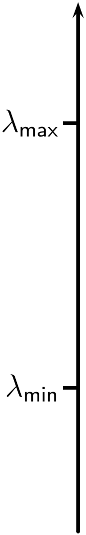
Properties:

- $\min\limits_{d \in D(k)} \underset{[d]}{\text{minb}}\, f \;\leq\; \min f \;\leq\; \min\limits_{d \in D(k)} \underset{[d]}{\text{minl}}\, f$
- $\mu\left(\left[\min\limits_{d \in D(k)} \underset{[d]}{\text{minb}}\, f,\; \min\limits_{d \in D(k)} \underset{[d]}{\text{minl}}\, f\right]\right) = \mathcal{O}\left(\frac{1}{k^2}\right)$

Bounds - in 2D



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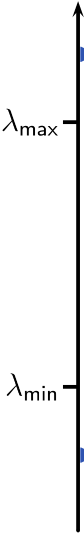
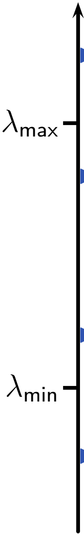


- $$\lambda_{\max}^{UB}(\mathbf{M}) = \max_{d \in D(k)} \frac{1}{2} \left[\underset{[d]}{\text{maxb}}\, q + \sqrt{\left(\underset{[d]}{\text{maxb}}\, s \right)^2 + \left(\underset{[d]}{\text{maxb}}\, t \right)^2} \right]$$

- $$\lambda_{\min}^{LB}(\mathbf{M}) = \min_{d \in D(k)} \frac{1}{2} \left[\underset{[d]}{\text{minb}}\, q - \sqrt{\left(\underset{[d]}{\text{maxb}}\, s \right)^2 + \left(\underset{[d]}{\text{maxb}}\, t \right)^2} \right]$$

Bounds - in 2D



- $$\lambda^{UB}_{\max}(\mathbf{M}) = \max_{d\in D(k)} \frac{1}{2}\left[\underset{[d]}{\text{maxb}}\, q + \sqrt{\left(\underset{[d]}{\text{maxb}}\, s\right)^2 + \left(\underset{[d]}{\text{maxb}}\, t\right)^2}\right]$$

- $$\lambda^{LB}_{\max}(\mathbf{M}) = \max_{d\in D(k)} \frac{1}{2}\left[\underset{[d]}{\text{maxl}}\, q + \sqrt{\left(\underset{[d]}{\text{maxl}}\, s\right)^2 + \left(\underset{[d]}{\text{maxl}}\, t\right)^2}\right]$$

- $$\lambda^{UB}_{\min}(\mathbf{M}) = \min_{d\in D(k)} \frac{1}{2}\left[\underset{[d]}{\text{minl}}\, q - \sqrt{\left(\underset{[d]}{\text{maxl}}\, s\right)^2 + \left(\underset{[d]}{\text{maxl}}\, t\right)^2}\right]$$

- $$\lambda^{LB}_{\min}(\mathbf{M}) = \min_{d\in D(k)} \frac{1}{2}\left[\underset{[d]}{\text{minb}}\, q - \sqrt{\left(\underset{[d]}{\text{maxb}}\, s\right)^2 + \left(\underset{[d]}{\text{maxb}}\, t\right)^2}\right]$$

Eigenvalues in 3D ?

Eigenvalues in 3D ? Yes

$$\lambda_i = \frac{||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,y}||^2 + ||\mathbf{X}_{,z}||^2}{3} + \frac{2}{\sqrt{6}}\, p(\cdot)\, \cos\left(\phi(\cdot) + i\,\frac{2\pi}{3}\right)$$

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$$p(\cdot) = \sqrt{\begin{array}{l} \left(\frac{2||\mathbf{X}_{,x}||^2}{3} - \frac{||\mathbf{X}_{,y}||^2 + ||\mathbf{X}_{,z}||^2}{3}\right)^2 + \left(\frac{2||\mathbf{X}_{,y}||^2}{3} - \frac{||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,z}||^2}{3}\right)^2 \\ + \left(\frac{2||\mathbf{X}_{,z}||^2}{3} - \frac{||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,y}||^2}{3}\right)^2 \\ +2\left[(\mathbf{X}_{,x} \cdot \mathbf{X}_{,y})^2 + (\mathbf{X}_{,x} \cdot \mathbf{X}_{,z})^2 + (\mathbf{X}_{,y} \cdot \mathbf{X}_{,z})^2\right] \end{array}}$$

Eigenvalues in 3D ? Yes

$$\lambda_i = \frac{||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,y}||^2 + ||\mathbf{X}_{,z}||^2}{3} + \frac{2}{\sqrt{6}}\, p(\cdot)\, \cos\left(\phi(\cdot) + i\frac{2\pi}{3}\right)$$

$$p(\cdot) = \sqrt{\begin{array}{l} \left(\frac{2||\mathbf{X}_{,x}||^2}{3} - \frac{||\mathbf{X}_{,y}||^2+||\mathbf{X}_{,z}||^2}{3}\right)^2 + \left(\frac{2||\mathbf{X}_{,y}||^2}{3} - \frac{||\mathbf{X}_{,x}||^2+||\mathbf{X}_{,z}||^2}{3}\right)^2 \\ + \left(\frac{2||\mathbf{X}_{,z}||^2}{3} - \frac{||\mathbf{X}_{,x}||^2+||\mathbf{X}_{,y}||^2}{3}\right)^2 \\ +2\left[(\mathbf{X}_{,x}\cdot\mathbf{X}_{,y})^2 + (\mathbf{X}_{,x}\cdot\mathbf{X}_{,z})^2 + (\mathbf{X}_{,y}\cdot\mathbf{X}_{,z})^2\right] \end{array}}$$

$\phi(\cdot)$ is awful

Eigenvalues in 3D ? Yes

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$\phi(\cdot)$ is awful

$$\left.\begin{array}{l}\lambda_{\text{max}} \\ \lambda_{\text{min}}\end{array}\right\} \simeq \frac{||\mathbf{X}_{,x}||^2 + ||\mathbf{X}_{,y}||^2 + ||\mathbf{X}_{,z}||^2}{3} \pm \frac{2}{\sqrt{6}}\, p(\cdot)$$